The Principles of Internal Ballistics.

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1. This paper contains the results of an investigation carried out at the Research Department, Royal Arsenal, Woolwich. It is published by permission of the War Office. The principal features are the methods of determination of the rates of burning of propellants and the treatment of the mathematical equations relating to the motion of the shot in the gun. No attempt has been made to improve on the methods of treating the distribution of motion and pressure through the gases in the gun, the resistances to the motion of the shot, the conduction of heat through the metal of the gun, or the recoil of the gun.

The general theory is sketched in terms of functions which require experimental evaluation; and then assumed provisional forms are assigned to these functions.

The author wishes to state that he owes the foundation of his knowledge of the subject, and also the opportunity of making these investigations, to Messrs. F. B. Pidduck and A. D. Crow.

General Theoretical Considerations.

2. We shall assume that the density and pressure of the propellant gases determine the temperature, so that we have always an equation of the form

$$T = H(\Delta, P), \tag{1}$$

where Δ denotes the density, P the pressure, and T the absolute temperature of the gases. We shall also assume that the chemical constitution of the gases is determined by the density and pressure, the attainment of chemical equilibrium being so rapid that the state appropriate to any pair of values of Δ and P is reached before these values have been sensibly departed from. We can then treat the gases arising from the burning of a propellant as a single body, whose state is defined by Δ and P, and all absorption or emission of heat due to changing chemical constitution may be allowed for in the specific heats taken.

Let

$$E(\Delta, P)$$

denote the intrinsic energy per unit mass of the gases when in the state given by Δ and P. By intrinsic energy we mean the maximum amount of work which the gases can be made to do by allowing them to expand to zero pressure and to cool to absolute zero of temperature. Let E denote the intrinsic energy per unit mass of the gases before they have done any work or lost any heat after being generated from the burning propellant. This we may call the intrinsic energy per unit mass of the unburnt propellant; it will depend only on the state of the unburnt propellant. The "uncooled pressure-density relation" will be

$$\mathrm{E}(\Delta, \mathrm{P}) = \mathrm{E}. \tag{2}$$

If we plot P against Δ, then the possible states of the gases will be given by all the points of an area, and (2) will be the equation of the upper boundary of this area.

Let σ_v and σ_p denote the specific heats at constant volume and constant pressure respectively; these will both be functions of Δ and P. If we write

$$v = 1/\Delta, \tag{3}$$

so that v is the volume per unit mass of the gases, we have, J denoting the mechanical equivalent of heat,

$$\left.\begin{aligned} \frac{\partial}{\partial \mathrm{P}}\mathrm{E}\left(\frac{1}{v}, \mathrm{P}\right) &= \mathrm{J}\sigma_v \frac{\partial}{\partial \mathrm{P}}\mathrm{H}\left(\frac{1}{v}, \mathrm{P}\right) \\ \frac{\partial}{\partial v}\mathrm{E}\left(\frac{1}{v}, \mathrm{P}\right) &= \mathrm{J}\sigma_p \frac{\partial}{\partial v}\mathrm{H}\left(\frac{1}{v}, \mathrm{P}\right) - \mathrm{P} \end{aligned}\right\}. \tag{4}$$

A necessary consequence of these is

$$\frac{\partial}{\partial v}\left\{\mathrm{J}\sigma_v \frac{\partial}{\partial \mathrm{P}}\mathrm{H}\left(\frac{1}{v}, \mathrm{P}\right)\right\} = \frac{\partial}{\partial \mathrm{P}}\left\{\mathrm{J}\sigma_p \frac{\partial}{\partial v}\mathrm{H}\left(\frac{1}{v}, \mathrm{P}\right) - \mathrm{P}\right\},$$

or

$$(\sigma_p - \sigma_v)\frac{\partial^2}{\partial v \partial \mathrm{P}}\mathrm{H}\left(\frac{1}{v}, \mathrm{P}\right) + \frac{\partial \sigma_p}{\partial \mathrm{P}}\frac{\partial}{\partial v}\mathrm{H}\left(\frac{1}{v}, \mathrm{P}\right) - \frac{\partial \sigma_v}{\partial v}\frac{\partial}{\partial \mathrm{P}}\mathrm{H}\left(\frac{1}{v}, \mathrm{P}\right) = \frac{1}{\mathrm{J}}. \tag{5}$$

Van der Waal's theoretical form for the function H(Δ, P) is

$$\mathrm{RH}(\Delta, \mathrm{P}) = (\mathrm{P} + a\Delta^2)(1/\Delta - b), \tag{6}$$

where R, a, b, are positive constants. If this law applied to the gases in question and if σ_v and σ_p were constants, we should have from (5)

$$\sigma_p - \sigma_v = \mathrm{R}/\mathrm{J}, \tag{7}$$

and then the integration of the equations (4) would give

$$(\gamma - 1)\mathrm{E}(\Delta, \mathrm{P}) = (\mathrm{P} + \gamma a \Delta^2)(1/\Delta - b), \tag{8}$$

where $\gamma = \sigma_p/\sigma_v$.

The uncooled density-pressure relation (2) would become

$$(\mathrm{P} + \gamma a \Delta^2)(1/\Delta - b) = \Pi, \tag{9}$$

where

$$\Pi = (\gamma - 1)\mathrm{E}. \tag{10}$$

The equation (9) may be written in the form

$$\frac{P}{\Delta} = bP + \frac{\Pi}{1+\gamma a \Delta^2/P}, \tag{11}$$

so that if we plot P/Δ against P, we obtain a curve which has for an asymptote

$$P/\Delta = bP + \Pi, \tag{12}$$

and always lies below this asymptote, except on the axis of $P = 0$, where the two meet. Equation (12) expresses Noble and Abel's pressure-density relation.

When a piece of homogeneous and isotropic propellant is surrounded by gas in a uniform state, we assume that combustion takes place by parallel layers. If u denotes the *rate of burning* down any normal, we shall suppose it determined by the state of the gases, so that

$$u = U(\Delta, P). \tag{13}$$

Let us consider an explosion in which all the constituent pieces of the charge are exactly alike, have been ignited all over their surfaces at exactly the same instant, and the state of the gases in contact with the unburnt charge is continuously uniform.

Let v_0 denote the volume and s_0 the total surface of a piece of propellant before ignition. At any time, t, during the explosion, let z denote the fraction of the charge burnt, and s the total surface of each piece. We shall have

$$s = s_0 \phi(z), \tag{14}$$

this defining $\phi(z)$, Charbonnier's form-function, and also

$$v_0\, dz/dt = us, \tag{15}$$

which give

$$\frac{dz}{dt} = \frac{s_0}{v_0} u \phi(z). \tag{16}$$

In ballistic applications we require the functions

$$E - E(\Delta, P), \qquad U(\Delta, P),$$

for all the values of Δ and P that occur. The function $H(\Delta, P)$ and the specific heats σ_v and σ_p may occur as auxiliaries in ballistic investigations, but, on the present assumptions, need not enter explicitly into practical ballistic calculations.

Closed Vessel Explosions.

3. Let us first consider an ideal closed vessel with non-conducting walls, and let us suppose that the attainment of mechanical equilibrium is so rapid that Δ and P may be taken as uniform through the vessel at any stage of the explosion.

Let Δ_1 and P_1 denote the maximum final values of Δ and P respectively. The relation between Δ and P will, throughout the explosion, be given by (2) and may be found from that between Δ_1 and P_1 in different explosions. If δ denote the density of the unburnt propellant, supposed uniform and constant throughout the explosion, we have

$$z = \frac{1/\Delta_1 - 1/\delta}{1/\Delta - 1/\delta}, \tag{17}$$

it being supposed that there is no non-gaseous residue after the explosion.

We suppose that the record gives P as a function of t; then (2) gives Δ and (17) z as functions of t. Using (16), we obtain u as a function of t, z, Δ or P. It is to be noted, however, that we have only considered here the determination of the function U (Δ, P) for the gases in the uncooled state; that is, only for the upper boundary of the representative area in the Δ, P plane over which the function is required.

If, after the explosion, we could draw off the heat from the gases so as to keep Δ_1 constant and Δ and P uniform through the vessel, and could take a record of the quantity of heat, Q, given off, as a function of P, we should have

$$JQ = E - E(\Delta_1, P), \tag{18}$$

so that $E - E(\Delta, P)$ could be determined for all desired states.

Now let us suppose that the release-valve of the vessel is connected with that of another vessel whose walls are conducting. Let both vessels be charged and the valve kept closed until the charge in the conducting vessel has been exploded and the gases allowed to cool. Now let the valve be opened, so that some of the cooled gases pass into the non-conducting vessel and reach a density Δ_0 and pressure P_0 round the unburnt charge of density Δ_1. Next let the valve be closed and the charge fired, and suppose it burns sufficiently slowly for all the gases in the vessel to be continuously in a uniform state. If this state be given by Δ, P, we shall have

$$z = \left(1 - \frac{\Delta_0}{\Delta}\right) \frac{1/\Delta_1 - 1/\delta}{1/\Delta - 1/\delta}, \tag{19}$$

$$\left(\frac{1}{\Delta} - \frac{1}{\delta}\right) \{E(\Delta, P) - E(\Delta_0, P_0)\} = \left(\frac{1}{\Delta_0} - \frac{1}{\Delta}\right) \{E - E(\Delta, P)\}. \tag{20}$$

When the function $E - E(\Delta, P)$ is known for all the values of Δ and P in question, the equation (20) gives the relation between Δ and P during the second explosion, the gases being now in a partially cooled state. The record for this second explosion is supposed to give P as a function of t; by means of (20) we obtain Δ as a function of t, and by means of (19) z as a function of t. On again using (16) we can then calculate the rate of burning, u, for states other than uncooled ones. A means is thus indicated for the determination of the function $U(\Delta, P)$ for all values of Δ and P required.

If we assume

$$E(\Delta, P) = \frac{1}{\gamma - 1} P\left(\frac{1}{\Delta} - \frac{1}{\delta}\right), \tag{21}$$

which is obtained from (8) by the usual practice of taking $a = 0$, $b = 1/\delta$, we obtain from (20)

$$P - P_0 = \frac{\Pi(1/\Delta_0 - 1/\Delta)}{(1/\Delta_0 - 1/\delta)(1/\Delta - 1/\delta)}, \tag{22}$$

as the relation between Δ and P during the second explosion.

Cooling.

4. So far we have imagined ideal conditions. When we come to consider explosions in actual vessels, the loss of heat by conduction through the walls is a serious factor. It is obvious that with the same density of loading in the same vessel the maximum pressure will decrease as the duration of the explosion increases, and the effect has been noticed ever since accurate experiments were made. It is also clear that for those shapes of pieces of propellant which make

$$\phi(z) \to 0, \qquad \text{as } z \to 1,$$

there will, at the instant of maximum pressure, be a certain fraction of the charge unburnt. For maximum pressure occurs when the rate of gain in pressure due to burning is balanced by the rate of loss in pressure due to cooling, and for the shapes in question the rate of gain by burning vanishes with the fraction of the charge unburnt. Further, for the same rate of loss by cooling, the fraction unburnt at maximum pressure will be greater the greater the duration of the explosion, or, practically, the greater the size of the piece of charge. The fall in maximum pressure from one explosion to another with the same density of loading, but with a larger size of piece of propellant, is therefore in excess of that representing a loss of energy through the walls of the vessel.

There exists, however, an important cooling phenomenon of a different type. This appears to have been first noticed by R. Blochmann* for high explosives. He found that for the same density of loading the pressure increased as the value of S/C decreased, S denoting the total internal surface of the explosion vessel and C its capacity, and he determined the limiting value of the pressure as $S/C \to 0$.

For propellants, the diminution of maximum pressure due to cooling had been elucidated in the Department. As was to be expected, it had been found that, for the same vessel and value of Δ_1, the value of P_1 increased as the duration of the explosion decreased. For each value of S/C and Δ_1 taken, the limiting value of P_1, as the duration tended to zero, was estimated. It was then found that, for the same value of Δ_1, these limiting values of P_1 increased as the values of S/C decreased, this being of the same nature as Blochmann's result. For each value of Δ_1 taken, second limiting values of P_1 as $S/C \to 0$ were found. These second limiting values of P_1 were taken to represent the results of an ideal explosion in which no cooling takes place, and from them the uncooled pressure-density relation (2) was constructed. Further experiments of the kind are urgently required, great attention being paid to the temperature conditions of the vessel and propellant, and to the regularity of size and shape in the pieces of propellant.

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* 'Dingler's Polytechnisches Journal, vol. 318, p. 234 (1903); the author owes this and many other references to Mr. F. B. Pidduck.

Apparent Rates of Burning.

5. During an actual explosion, we do not know the true relation between Δ and P, and so z cannot be found accurately as a function of t for the purpose of calculating the rate of burning by (16).

The method indicated at the beginning of § 3, when applied as if no cooling took place, is practically that established by Vieille,* and hitherto, in no available publication on rates of burning, have cooling effects been eliminated or allowed for. Previous to Vieille's work, writers on ballistics had usually taken

$$u = u'P^{\alpha}, \tag{23}$$

u' and α being constants. For α different writers had used

$$0,\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{5},\ \tfrac{2}{3},\ 1.$$

Vieille decided on the same law, as an approximation, for the propellants with which he experimented, and took for α

$$\tfrac{2}{3},\ \tfrac{5}{9},\ \tfrac{3}{5},$$

for nitrocellulose, cordite, and ballistite respectively.

For all the French propellants, Charbonnier, Gossot and Liouville claim the validity of (23), but whereas Charbonnier† maintains that $\alpha = 1$, Gossot and Liouville‡ insist that $\alpha = \frac{2}{3}$. For cordite M.D., heated to 80° F. before ignition, Mansell§ found the relation

$$u = u_0 + u_1 P, \tag{24}$$

u_0 and u_1 being constants.

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* 'Mémorial des Poudres et Salpêtres,' vol. 6 (1893).

† See, *e.g.*, 'Balistique Intérieure,' p. 76 (Paris, 1908).

‡ See 'Mémorial des Poudres et Salpêtres,' vol. 13 (1906).

§ 'Phil. Trans.,' A, vol. 207, pp. 243–262 (1907).

Both (23) and (24) make u independent of Δ, and it is impossible to obtain u as a function of more than one independent variable by the methods that have been used. By these methods, each record leads to a relation between u and P, which, when the duration of the explosion is not too long, is an approximation to the true relation with Δ connected to P according to (2).

It is clear, however, that as for the same value of P in different explosions we may have different values of Δ, we may expect the derived relation between u and P to vary from the analysis of one record to that of another. On the one hand, the relation between Δ and P cannot always be that assumed in the analysis, and, on the other hand, the true rate of burning may vary with the value of Δ for the same value of P.

A first step in the study of true rates of burning is the elimination of cooling effects during the whole of the explosion. This requires an elaborate series of experiments, such as those already referred to. When the customary methods are used, we refer to the results as "apparent."

An apparent rate of burning is stated as a function of the pressure only; this function varies with the circumstances of the explosion, such as the size and shape of the vessel, the size of the pieces of propellant, the density of the charge, the temperature of the vessel and of the charge.

Uncooled Rate of Burning.

6. We shall now show how to apply to the whole of the pressure-time records a double limiting process, so as to eliminate cooling effects.

We still make all the assumptions stated at the beginning of § 3, except that relating to cooling. From each record we calculate $\dot{y}v_0/s_0$ as a fraction of $y = \mathrm{P}/\mathrm{P}_1$. In the absence of cooling effects, this relation would depend only on Δ_1 and be independent of v_0/s_0 and S/C. On this account, and also because it obviates the uncertainty of corresponding times in different explosions, it is this relation between $\dot{y}v_0/s_0$ and y, and not that between P and t, which is subjected to the limiting process.

We suppose that we have pressure-time records for each of m values of v_0/s_0 in closed vessels having n values of S/C. For each value of S/C, and each of the values

$$0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}3,\ \ldots\ 0{\cdot}9,\ 1,$$

of y, $\dot{y}v_0/s_0$ is plotted against the values of v_0/s_0.

From each set of m points thus obtained, the limiting value of $\dot{y}v_0/s_0$ as $v_0/s_0 \rightarrow 0$ is estimated. We are thus enabled to plot the limiting relation between $\dot{y}v_0/s_0$ and y for each value of S/C.

Again for each of the values

$$0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}3, \dots\ 0{\cdot}9,\ 1$$

of y, the limiting values of $\dot{y}v_0/s_0$ already found are plotted against the corresponding values of S/C. From each of the set of n points thus obtained, the second limiting value of $\dot{y}v_0/s_0$ as $S/C \rightarrow 0$ is estimated.

We are thus able to plot the double limiting relation between $\dot{y}v_0/s_0$ and y, and this is taken as corresponding to an explosion in which there are no cooling effects. We now suppose that our symbols refer to this hypothetical explosion, and use the method sketched at the beginning of § 3.

From (17) z can be calculated as a function of y, since Δ is connected with P by the "uncooled" relation. From (16) we have

$$u = \dot{y}\frac{v_0}{s_0}\frac{1}{\phi(z)}\frac{dz}{dy}, \tag{25}$$

from which u can be calculated as a function of y, and therefore of P. This represents the "uncooled" rate of burning function, *i.e.*, the function (13), when Δ and P are connected by (2).

Vessel of Variable Capacity.[*]

7. Let unit charge be placed in a vessel with non-conducting walls whose capacity is variable, and let z denote the fraction burnt when the capacity is v. Then we have

$$\Delta = \frac{z}{v-(1-z)/\delta},$$

still regarding δ as uniform and constant.

The total intrinsic energy in the vessel is

$$(1-z)\mathrm{E}+z\mathrm{E}(\Delta, \mathrm{P}),$$

and we deduce

$$\frac{d}{dv}z\{\mathrm{E}(\Delta, \mathrm{P})-\mathrm{E}\}+\mathrm{P} = 0. \tag{26}$$

This is the fundamental differential equation for the circumstances we are now considering, and if v be interpreted as the capacity of the vessel per unit mass of the charge, it is of general application. It is an equation between v, z and P which cannot be integrated without further relation between these variables.

The non-integrability of the equation (26) may be seen from the consideration that the change in the state of the gases for given changes in capacity and amount burnt, must depend on the way these changes have been made, and not simply on the extent of the changes. If an expansion takes place with a small fraction burnt, a small amount of work is done by the low pressure; if the same expansion takes place with a large fraction burnt, a large amount of work is done by the high pressure. In these two instances the final intrinsic energies of the gases must be different, though the capacities and fractions burnt be the same.

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[*] *Cf.* A. G. Hadcock, 'Roy. Soc. Proc.,' A, vol. 94, pp. 484-488 (1917).

In the absence of experimental knowledge of the function E(Δ, P) we are again forced on the provisional assumptions. Assuming the formula (21) we have

$$zE(\Delta, P) = \frac{1}{\gamma-1}P\left(v-\frac{1}{\delta}\right), \tag{27}$$

since

$$\frac{1}{\Delta}-\frac{1}{\delta} = \frac{1}{z}\left(v-\frac{1-z}{\delta}\right)-\frac{1}{\delta} = \frac{1}{z}\left(v-\frac{1}{\delta}\right). \tag{28}$$

On using (10) the equation (26) becomes

$$\frac{1}{\gamma-1}\frac{d}{dv}\left\{P\left(v-\frac{1}{\delta}\right)-z\Pi\right\}+P = 0, \tag{29}$$

or

$$\Pi\frac{dz}{dv} = \left(v-\frac{1}{\delta}\right)\frac{dP}{dv}+\gamma P. \tag{30}$$

Equations of the Gun.

8. Let C now denote the capacity of the chamber of the gun, A the area of bore, and x the travel up to time t; let m denote the effective mass of the shot, c that of the charge, and z the fraction burnt up to time, t; also, let V denote the velocity of the shot and P the pressure of the gases at the time t. Then we have

$$v = (C+Ax)/c, \tag{31}$$

$$V = dx/dt, \tag{32}$$

and

$$m\,dV/dt = (1-\beta)A(P-R), \tag{33}$$

where R and β are inserted to allow for resistance and the angular acceleration of the shot. The equation (26) yields

$$\frac{c}{A}\frac{d}{dx}z\{E(\Delta, P)-kE\}+P = 0, \tag{34}$$

where

$$\Delta = \frac{z}{(C+Ax)/c-(1-z)/\delta}, \tag{35}$$

and k is inserted to allow for loss of heat through the walls of the gun.

The equations (32), (33), (34) and (16), we shall refer to as the "equations of the gun." Equations (33) and (34) give

$$m\frac{d}{dx}(\tfrac{1}{2}V^2)+(1-\beta)c\frac{d}{dx}z\{E(\Delta, P)-kE\}+(1-\beta)AR = 0, \tag{36}$$

which, on treating β as a constant, yields

$$\tfrac{1}{2}\frac{m}{1-\beta}V^2 + cz\{E(\Delta, P) - kE\} + A\int_0^x R\,dx = 0, \tag{37}$$

using the conditions that when $x = 0$, $V = 0$, and $E(\Delta, P) = kE$.

The equation (37) is the *energy integral*, and may be used instead of (33) and (34) in the equations of the gun.

We now again fall back on the provisional assumptions, and use the equation (30) and the law (24) for the rate of burning function. Equations (34) and (16) may therefore be written, on treating k as constant,

$$kc\Pi\frac{dz}{dx} = \left(C + Ax - \frac{c}{\delta}\right)\frac{dP}{dx} + \gamma AP, \tag{38}$$

$$\frac{dz}{dt} = (u_0 + u_1 P)\frac{s_0}{v_0}\phi(z), \tag{39}$$

the initial conditions being that motion begins at $x = 0$ when $P = R_0$, the value of R at $x = 0$. Up to this stage we have

$$k\Pi z = (C/c - 1/\delta)P. \tag{40}$$

When *maximum pressure* occurs during burning, we have from (38)

$$kc\Pi\,dz/dx = \gamma AP,$$

and therefore, from (32) and (39)

$$\frac{V'}{\phi(z')} = \frac{kc\Pi}{\gamma A}\frac{s_0}{v_0}(u_1 + u_0/P'), \tag{41}$$

on using accents to denote evaluations under the circumstances of maximum pressure.

Dimensional Relations.

9. Take

$$\theta = 1 + \frac{Ax}{C - c/\delta}, \tag{42}$$

$$P = \frac{kc\Pi}{C - c/\delta}\cdot p, \tag{45}$$

$$M = \frac{(1-\beta)A^2 v_0^2}{kc\Pi m u_1^2 s_0^2}, \tag{43}$$

$$R = \frac{kc\Pi}{C - c/\delta}\cdot r, \tag{46}$$

$$V = (1-\beta)\frac{Av_0}{mu_1 s_0}\cdot v, \tag{44}$$

$$\varpi = \frac{C - c/\delta}{kc\Pi}\cdot\frac{u_0}{u_1}, \tag{47}$$

and for the time instead of t take

$$\frac{v_0}{k\Pi u_1 s_0}\left(\frac{C}{c} - \frac{1}{\delta}\right)\cdot t. \tag{48}$$

Then the equations of the gun may be written

$$d\theta/dt = Mv, \tag{49}$$

$$dz/d\theta = \theta\,dp/d\theta + \gamma p, \tag{51}$$

$$dv/dt = p - r, \tag{50}$$

$$dz/dt = (p + \varpi)\phi(z), \tag{52}$$

the initial conditions being that

$$\theta = 1, \qquad v = 0, \qquad p = z = r_0, \tag{53}$$

while the *energy integral* becomes

$$\frac{1}{\gamma-1}(z-p\theta) = \tfrac{1}{2}\mathrm{M}v^2+\int_1^\theta r\,d\theta, \tag{54}$$

and the condition for *maximum pressure* when it occurs during burning

$$\frac{v'}{\phi(z')} = \frac{1}{\gamma\mathrm{M}}(1+\varpi/p'). \tag{55}$$

The solution of the equations of the gun is then seen to depend on the constants M and ϖ, and on the functions $\phi(z)$ and r. These may remain the same over a considerable variety of circumstances from the point of view of applications, so that a gun calculation for one set of circumstances may be made to yield the solution for certain other sets of circumstances. This principle provides the proof of the *general theorem* immediately following.

If $\qquad \mathrm{A},\ \mathrm{C}-c/\delta,\ x,\ m,\ c,\ v_0/s_0,\ k\Pi,\ u_1,\ u_0,\ 1-\beta,$

be multiplied by a parameter raised respectively to the powers

$$(\mathrm{A}),\ (\mathrm{C}),\ (x),\ (m),\ (c),\ (v_0),\ (\Pi),\ (u_1),\ (u_0),\ (\beta),$$

then V will be multiplied by the parameter raised to the power

$$(\mathrm{A})+(v_0)+(\beta)-(m)-(u_1),$$

P will be multiplied by the parameter raised to the power

$$(c)+(\Pi)-(\mathrm{C}),$$

time will be multiplied by the parameter raised to the power

$$(v_0)+(\mathrm{C})-(u_1)-(\Pi)-(c),$$

providing

$$(\mathrm{A})+(x) = (\mathrm{C}),$$

$$(c)+(m)+2(u_1)+(\Pi) = 2(\mathrm{A})+2(v_0)+(\beta)$$

$$(\Pi)+(c)+(u_1) = (\mathrm{C})+(u_0),$$

and R be changed like P.

Certain special cases of this theorem have long been used; thus take

$$(\mathrm{A}) = 2,\quad (\mathrm{C}) = 3,\quad (x) = 1,\quad (m) = 3,\quad (c) = 3,\quad (v_0) = 1,$$

$$(\Pi) = (u_1) = (u_0) = (\beta) = 0,$$

and we have the theorem that similar guns similarly charged with the same propellant produce equal pressures and equal velocities in linearly similar times.

General Differential Equation.

10. The equations (49), (50), and (52) give

$$v = \frac{1}{\mathrm{M}} \frac{d\theta}{dt}, \tag{56}$$

$$p = r + \frac{1}{\mathrm{M}} \frac{d^2\theta}{dt^2}, \tag{57}$$

$$p + \varpi = \frac{d}{dt} \int_{r_0}^{z} \frac{dz}{\phi(z)}, \tag{58}$$

and then on taking r *constant during band engraving*, we obtain

$$\int_{r_0}^{z} \frac{dz}{\phi(z)} = \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + (\varpi + r_0)\, t. \tag{59}$$

On solving this equation for z we have, say,

$$z = \mathrm{Z}\left\{ r_0, \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + (\varpi + r_0)\, t \right\}, \tag{60}$$

where the function Z depends on the function $\phi(z)$. Substituting in (54) we obtain, on rearrangement,

$$\theta \frac{d^2\theta}{dt^2} + \tfrac{1}{2}(\gamma - 1)\left(\frac{d\theta}{dt}\right)^2 - \mathrm{MZ}\left\{ r_0, \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + (\varpi + r_0)\, t \right\} + \gamma \mathrm{M} r_0 \theta = (\gamma - 1)\, \mathrm{M} r_0, \tag{61}$$

which is a differential equation for θ in terms of t.

For pieces of propellant with a *constant burning surface*, so that $\phi(z) = 1$, we have

$$\mathrm{Z}\left\{ r_0, \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + (\varpi + r_0)\, t \right\} = r_0 + \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + (\varpi + r_0)\, t, \tag{62}$$

and the differential equation (61) becomes

$$\theta \frac{d^2\theta}{dt^2} + \tfrac{1}{2}(\gamma - 1)\left(\frac{d\theta}{dt}\right)^2 - \frac{d\theta}{dt} + \gamma \mathrm{M} r_0 (\theta - 1) = \mathrm{M}(\varpi + r_0)\, t. \tag{63}$$

After band engraving, we take $r = 0$ and obtain

$$\int_{z_1}^{z} \frac{dz}{\phi(z)} = \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + \varpi t - (v_1 + \varpi t_1), \tag{64}$$

where the suffix 1 refers to the end of band engraving. This gives

$$z = \mathrm{Z}\left\{ z_1, \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + \varpi t - (v_1 + \varpi t_1) \right\}, \tag{65}$$

and on substituting in (54) we obtain, on rearrangement,

$$\theta \frac{d^2\theta}{dt^2} + \tfrac{1}{2}(\gamma - 1)\left(\frac{d\theta}{dt}\right)^2 - \mathrm{MZ}\left\{ z_1, \frac{1}{\mathrm{M}} \frac{d\theta}{dt} + \varpi t - (v_1 + \varpi t_1) \right\} + (\gamma - 1)\, \mathrm{M} r_0 (\theta_1 - 1) = 0. \tag{66}$$

Again, for pieces of propellant with a *constant burning surface*, this differential equation becomes

$$\theta\frac{d^2\theta}{dt^2}+\tfrac{1}{2}(\gamma-1)\left(\frac{d\theta}{dt}\right)^2-\frac{d\theta}{dt} = \mathrm{M}\varpi t+\mathrm{M}\left\{z_1-(v_1+\varpi t_1)-(\gamma-1)r_0(\theta_1-1)\right\}. \tag{67}$$

When $\varpi = 0$ so that the rate of burning of the propellant is proportional to the pressure, the method of the present section is not the most direct, and when $\varpi \neq 0$ so that the rate of burning is not proportional to the pressure, the differential equations we have obtained cannot be integrated in terms of elementary functions. Of course, in any particular case we may obtain a numerical solution to any required degree of accuracy, by the methods of step-by-step integration, and in this way tables could be constructed giving the solution in terms of M, ϖ and r_0 for each form of $\phi(z)$. But the transformations of the present section make a contribution to the results of the next section.

Concentrated Band, Characteristic Parameters.

11. By concentrated driving band we mean the conditions presented by

$$r = 0, \quad \text{for } \theta > 1\,; \qquad r_0 \neq 0,$$

which is Charbonnier's method of treating band engraving.

The solution of the equations of the gun is then determined completely by the function $\phi(z)$ and the values of

$$\mathrm{M}, \quad \varpi, \quad r_0, \quad \theta_3, \tag{68}$$

the suffix 3 referring to the muzzle. In particular, the value of v_3 and whether the charge is all burnt or not, depend only on these. If the charge is all burnt in the gun, then whether the maximum pressure occurs while it is burning or not, depends only on $\phi(z)$ and the values of

$$\mathrm{M}, \quad \varpi, \quad r_0, \tag{69}$$

as does also the value of p'.

For any one form of the function $\phi(z)$ the functional relations here indicated are the same for all guns and propellants to which the equations (33), (38), (39) apply.

When $\phi(z) = 1$, and maximum pressure occurs during burning, we see from (67) that $\mathrm{M}p'$ depends only on $\mathrm{M}\varpi$ and $\mathrm{M}r_0$, since for a concentrated band

$$\theta_1 = 1, \quad v_1 = 0, \quad z_1 = r_0.$$

Reverting now to the notation of § 8, we see that the solution of the equations of the gun, and in particular the value of

$$\frac{mu_1 s_0}{(1-\beta)\,\mathrm{A}v_0}\,\mathrm{V}_3, \tag{70}$$

depend only on

$$\frac{(1-\beta)A^2v_0^2}{kc\Pi m u_1^2 s_0^2}, \qquad \frac{(C-c/\delta)u_0}{kc\Pi u_1}, \qquad \frac{C-c/\delta}{kc\Pi}R_0, \qquad \frac{C_3-c/\delta}{C-c/\delta}, \tag*{(71)}$$

C_3 denoting the total capacity of the gun.

For the same propellant whose rate of burning is not proportional to the pressure, these may be replaced respectively by

$$\frac{ms_0}{(1-\beta)Av_0}\cdot V_3, \tag*{(70)′}$$

and

$$\frac{(1-\beta)A^2v_0^2}{kcms_0^2}, \qquad \frac{C-c/\delta}{kc}, \qquad R_0, \qquad \frac{C_3-c/\delta}{C-c/\delta}. \tag*{(71)′}$$

When the charge is all burnt in the gun, then whether maximum pressure occurs during burning or not, and also the value of

$$\frac{C-c/\delta}{kc\Pi}P', \tag*{(72)}$$

depend only on the values of

$$\frac{(1-\beta)A^2v_0^2}{kc\Pi m u_1^2 s_0^2}, \qquad \frac{(C-c/\delta)u_0}{ke\Pi u_1}, \qquad \frac{C-c/\delta}{kc\Pi}R_0. \tag*{(73)}$$

Again, for the same propellant whose rate of burning is not proportional to the pressure, these may be replaced by

$$P' \tag*{(72)′}$$

and

$$\frac{(1-\beta)A^2v_0^2}{kcms_0^2}, \qquad \frac{C-c/\delta}{kc}, \qquad R_0, \tag*{(73)′}$$

respectively.

If, further, the pieces of propellant have a constant burning surface, and maximum pressure occurs during burning, then

$$\frac{(1-\beta)(C-c/\delta)}{m}\left(\frac{Av_0}{kc\Pi u_1 s_0}\right)^2 P', \tag*{(74)}$$

depends only on

$$\frac{(1-\beta)(C-c/\delta)}{m}\left(\frac{Av_0}{kc\Pi u_1 s_0}\right)^2 \frac{u_0}{u_1}, \qquad \frac{(1-\beta)(C-c/\delta)}{m}\left(\frac{Av_0}{kc\Pi u_1 s_0}\right)^2 R_0. \tag*{(75)}$$

One of the relations of this section has been used by F. B. Pidduck. Taking the same propellant with the rate of burning proportional to the pressure, so that $u_0 = 0$, we have that

$$\frac{(C-c/\delta)}{kc}P', \tag*{(72)″}$$

depends only on

$$\frac{(1-\beta)A^2v_0^2}{kcms_0^2}, \qquad \frac{(C-c/\delta)}{kc}R_0. \tag*{(73)″}$$

Pidduck plotted observed values of the first of these three quantities against the second, and sought the effect of the third.

The method appears capable of much further utility. For example, on taking the same propellant and

$$\phi(z) = 1, \qquad u_0 = 0,$$

we have, when maximum pressure occurs during burning

$$\frac{(1-\beta)(C-c/\delta)}{m}\left(\frac{Av_0}{kcs_0}\right)^2 P', \tag{74)'}$$

depending only on

$$\frac{(1-\beta)(C-c/\delta)}{m}\left(\frac{Av_0}{kcs_0}\right)^2 R_0. \tag{75)'}$$

But the greatest value of the present relations is when the rate of burning is not proportional to the pressure, and the equations of the gun cannot be integrated by elementary functions. For example, when with the same propellant

$$\phi(z) = 1, \qquad u_0 \neq 0,$$

and when the maximum pressure occurs during burning, P′ depends only on

$$\frac{(1-\beta)(C-c/\delta)}{m}\left(\frac{Av_0}{kcs_0}\right)^2, \quad R_0. \tag{75)''}$$

After Engraving with Rate of Burning Proportional to Pressure.

12. We shall now take $r = 0$, $\varpi = 0$, and start from the conditions

$$\theta = \theta_1, \quad z = z_1, \quad v = v_1, \quad p = p_1.$$

From (50) and (52) we have

$$dv/dz = 1/\phi(z), \tag{76}$$

so that

$$v = v_1 + \int_{z_1}^{z} \frac{dz}{\phi(z)}, \tag{77}$$

and from (49) and (52) we have

$$p = \frac{Mv}{\phi(z)}\frac{dz}{d\theta}. \tag{78}$$

On substituting in (54) from (77) and (78) we obtain

$$\tfrac{1}{2}(\gamma-1)\frac{1}{\theta}\frac{d\theta}{dz} = \frac{v_1 + \displaystyle\int_{z_1}^{z}\frac{dz}{\phi(z)}}{\dfrac{2}{M}\left\{\dfrac{z}{\gamma-1} - \displaystyle\int_1^{\theta_1} r\,d\theta\right\} - \left\{v_1 + \displaystyle\int_{z_1}^{z}\frac{dz}{\phi(z)}\right\}^2} \cdot \frac{1}{\phi(z)},$$

giving

$$\log\left(\frac{\theta}{\theta_1}\right)^{\frac{1}{2}(\gamma-1)} = \int_{z_1}^{z} \frac{v_1 + \displaystyle\int_{z_1}^{z}\frac{dz}{\phi(z)}}{\dfrac{2}{M}\left\{\dfrac{z}{\gamma-1} - \displaystyle\int_1^{\theta_1} r\,d\theta\right\} - \left\{v_1 + \displaystyle\int_{z_1}^{z}\frac{dz}{\phi(z)}\right\}^2} \cdot \frac{dz}{\phi(z)}, \tag{79}$$

which is the general θ, z relation after engraving and during burning.

For a concentrated band (77) and (79) reduce to

$$v = \int_{r_0}^{z} \frac{dz}{\phi(z)}, \tag{77$'$}$$

and

$$\log \theta^{\frac{1}{2}(\gamma-1)} = \int_{r_0}^{z} \frac{\displaystyle\int_{r_0}^{z} \frac{dz}{\phi(z)}}{\dfrac{2z}{(\gamma-1)\mathrm{M}} - \left\{\displaystyle\int_{r_0}^{z} \frac{dz}{\phi(z)}\right\}^2} \cdot \frac{dz}{\phi(z)}. \tag{79$'$}$$

The equations (77)$'$ and (79)$'$ form the basis of Charbonnier's solution, but in evaluating (79)$'$ he approximates except for $\phi(z) = 1$.

On evaluating (79) for

$$\phi(z) = 1, \qquad \phi(z) = (1-z)^{\frac{1}{2}},$$

we obtain

$$\frac{\theta}{\theta_1} = \left\{\frac{b+2c\zeta_1-\zeta_1^2}{b+2c\zeta-\zeta^2}\right\}^{\lambda} \left\{\frac{(a-c+\zeta_1)(a+c-\zeta)}{(a-c+\zeta)(a+c-\zeta_1)}\right\}^{\mu}, \tag{80}$$

with

$$a = (b+c^2)^{\frac{1}{2}},$$

where for $\phi(z) = 1$, or a constant burning surface,

$$\left.\begin{aligned} \zeta &= z, \quad \zeta_1 = z_1, \\ b &= -\frac{2}{\mathrm{M}}\int_1^{\theta_1} r\,d\theta - (z_1 - v_1)^2 \\ c &= 1/(\gamma-1)\mathrm{M} + (z_1 - v_1) \\ \lambda &= 1/(\gamma-1) \\ \mu &= -1/(\gamma-1)^2\mathrm{M}a \end{aligned}\right\}, \tag{81}$$

and for $\phi(z) = (1-z)^{\frac{1}{2}}$, or cylindrical pieces of propellant,

$$\left.\begin{aligned} \zeta &= (1-z)^{\frac{1}{2}}, \qquad \zeta_1 = (1-z_1)^{\frac{1}{2}} \\ b &= \frac{1}{1+1/2(\gamma-1)\mathrm{M}}\left\{\frac{1}{2\mathrm{M}}\left(\frac{1}{\gamma-1} - \int_1^{\theta_1} r\,d\theta\right) - (\zeta_1 + \tfrac{1}{2}v_1)^2\right\} \\ c &= \frac{\zeta_1 + \frac{1}{2}v_1}{1+1/2(\gamma-1)\mathrm{M}} \\ \lambda &= \frac{1}{\gamma-1+1/2\mathrm{M}} \\ \mu &= \frac{2\mathrm{M}}{a}\,\frac{\zeta_1 + \frac{1}{2}v_1}{\{1+2(\gamma-1)\mathrm{M}\}^2} \end{aligned}\right\}. \tag{82}$$

After all the charge is burnt, our equations reduce to those in general use.

The result for a concentrated band may be obtained very easily by putting

$$\theta = 1, \qquad v_1 = 0, \qquad z_1 = r_0\,(\neq 0),$$

in (81) and (82), but the derivation of those for *no band* requires a limiting process, so that they are most easily obtained directly.

For $\phi(z) = 1$ the results are given by Charbonnier,* while for $\phi(z) = (1-z)^{\frac{1}{2}}$ we have

$$v = 2\{1-(1-z)^{\frac{1}{2}}\}, \tag{83}$$

$$\theta = [\tfrac{1}{2}-(\gamma-1)\mathrm{M}+\{\tfrac{1}{2}+(\gamma-1)\mathrm{M}\}(1-z)^{\frac{1}{2}}]^{-\frac{1}{\frac{1}{2}(\gamma-1)+1/4\mathrm{M}}}, \tag{84}$$

$$p\theta = [z-2(\gamma-1)\mathrm{M}\{1-(1-z)^{\frac{1}{2}}\}]. \tag{85}$$

The Harmonic Development of the Tide-generating Potential.

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§ 1. *Introduction.*

The harmonic development of the tide-generating potential is the basis of most work on tidal observations, and since 1883 the development given by Sir G. H. Darwin has been universally used and has been of remarkable value. But discrepancies between prediction and observation are serious and have been attributed to faulty "harmonic constants"; it has been assumed that if these were improved better predictions would be obtained, and it has also been tacitly assumed that it is only necessary to consider the harmonic constituents as given by Darwin. Recent work, however, especially at the Tidal Institute, has shown that when all the "Darwinian constituents" are removed from the tidal height there is a residue composed of constituents which are not included in his schedules. These are such that any slight improvements possible in the "constants" usually obtained are comparatively negligible.

The obvious course, therefore, was to make a more thorough development of the potential, and in view of the unknown nature of the residues great accuracy was obviously desirable, especially as the possibility of resonance has always to be considered. The development given in this paper, even if it proves to be needlessly thorough for practical tidal work, will cover the needs of research work, since it includes all terms whose coefficients (relatively to the greatest coefficient) are greater than 0·00010.

Darwin used the old lunar theory and referred everything to the orbit rather than to the ecliptic; his results are all given in the algebraic form, arithmetic being used only to decide what terms to omit. His development is a quasi-harmonic development because he retains factors in the coefficients

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* *Loc. cit.*, Chap. VII.